

Abstract

In this lab we obtained experimental measurements of water height as a function of time of a draining-tanks apparatus and compared the measured response data to modeled responses to estimate a discharge coefficient for both tanks in the apparatus. The accuracy of our estimate was determined graphically and quantitatively. We then validated the single-tank model results using the results of our two-tank experiment. This experiment was run to strengthen the connection between the theoretical models presented in lectures and the actual behavior of tank draining behavior, devices, processes and systems which is accomplished through observation and experimentation. This report presents our experimental theory, model, analysis, results, and reflections. Our model for both the upper and lower tanks closely fit the original gathered data which was analyzed both graphically and quantitatively except for certain parts. Our results from the two-tank experiment were similar to our single-tank model results allowing us to validate them both. Tanks and vessels are used extensively in the process industry for both batch and continuous processes and the results from this experiment could possibly contribute to an industry application.

Introduction

The drainage of a two-tank system was modeled using conservation of mass and orifice relationships, as well as traditional simulation diagram techniques. The purpose of this exercise was to compare the physical and model responses of the water height within each tank independently, and the system as a whole.

This lab was conducted by a group of students remotely, so raw data was collected and provided prior to analysis. Data was collected through the use of pressure transducers within each tank, then analyzed after converting the voltage outputs of each sensor to meaningful height vs. time data. In addition, the water height and time relationship was modeled theoretically to validate the team's experimental findings. In order to predict the behavior of the system as a whole, each individual tank must be analyzed to determine a discharge coefficient which most appropriately describes its behavior. To do this experimentally, standard error estimate values were optimized to achieve the most accurate discharge coefficient possible. Then the team used these coefficients, as well as other physical properties of each tank and system environment to accurately refine and validate a model which captures the behavior of the experiment.

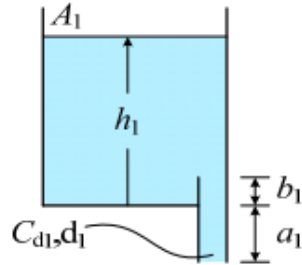
The overall objectives of the experiment were to describe the process of calibrating the tank pressure transducers and operate the apparatus to obtain experimental measurements of water height as a function of time. By the end of the exercise, the experimental and model responses were used to obtain an accurate discharge coefficient, then validated single-tank models using a two-tank experiment. The process of reaching our conclusions is documented through a recapitulation of our method of analysis and a thorough discussion of our results and associated learning.

Method of Analysis

To build up to the complex two tank system, the tanks are first modeled and simulated individually.

Below is shown the schematic of both tanks.

Tank 1:



Tank 2:

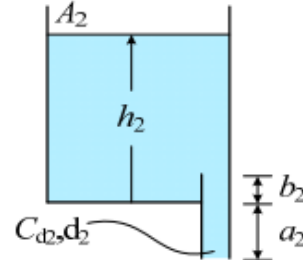
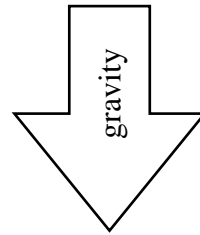
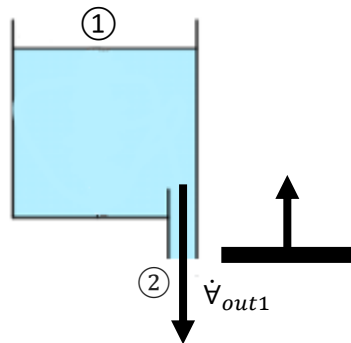


Figure 1: side by side schematic of each tank present in the system.

Shown above are the two tanks, the key to the analysis is the use of conservation principles. Specifically, in this case the application of conservation of mass rate form and conservation of energy finite form. A_1 and A_2 are the tanks' respective surface areas. b_1 and b_2 are the tanks' respective inner wall heights. a_1 and a_2 are the tank's respective drain pipe lengths. d_1 and d_2 are the tank's respective outlet diameters. C_{d1} and C_{d2} are the tanks' respective orifice discharge coefficients. Before applying any conservation principles, the free body diagrams must be defined for each tank.

Tank 1 FBD:



Tank 2 FBD:

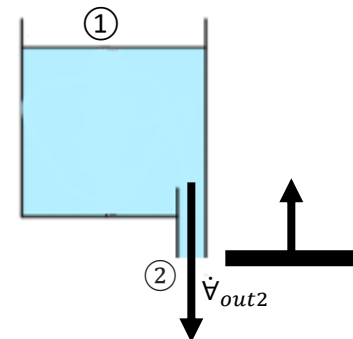


Figure 2: side by side free body diagram of each tank present in the system.

$$A(h) \frac{dh}{dt} = \dot{V}_{in} - \dot{V}_{out} \quad (1.1)$$

Shown above is the incompressible flow version of the rate form of conservation of mass equation. Applying this equation to the free body diagrams yields the following (tank 1 then tank 2).

$$A_1 \cdot \frac{dh}{dt} = -\dot{V}_{out1} \quad (1.2)$$

$$A_2 \cdot \frac{dh}{dt} = -\dot{V}_{out2} \quad (1.3)$$

Where $\frac{dh}{dt}$ is the change in height of the water level of the tank with respect to time. Using the orifice volume flow rate equation which is derived from the modified Bernoulli equation, a formula for \dot{V}_{out} can be derived.

$$\dot{V}_{out} = A_{out} \cdot C_d \cdot \sqrt{\frac{2}{\rho} \cdot ((P_1 - P_2) + \rho \cdot g \cdot (z_1 - z_2))} \quad (2.1)$$

The pressure at point 1 and point 2, P_1 and P_2 respectively are equal. ρ is the density of water. Lastly, z_1 and z_2 are the heights of each Bernoulli point relative to the datum. The datum in this scenario is the bottom of the output pipe, not the bottom of the tank. The reduced orifice volume flow rate equation is shown below for tank 1 and tank 2 respectively.

$$\dot{V}_{out1} = A_{out1} \cdot C_{d1} \cdot \sqrt{2 \cdot g \cdot (z_{1,tank1} - z_{2,tank1})} \quad (2.2)$$

$$\dot{V}_{out2} = A_{out2} \cdot C_{d2} \cdot \sqrt{2 \cdot g \cdot (z_{1,tank2} - z_{2,tank2})} \quad (2.3)$$

Next equations 1.1 and 1.2 are combined with 2.2 and 2.3 to form the model equations for both tanks.

$$A_1 \cdot \frac{dh_1}{dt} = -A_{out1} \cdot C_{d1} \cdot \sqrt{2 \cdot g \cdot (z_{1,tank1} - z_{2,tank1})} \quad (3.1)$$

$$A_2 \cdot \frac{dh_2}{dt} = -A_{out2} \cdot C_{d2} \cdot \sqrt{2 \cdot g \cdot (z_{1,tank2} - z_{2,tank2})} \quad (3.2)$$

To simplify these equations, further substitutions of known variables for the distances are necessary. Due to the location of the datum, a height must be added anytime the distance relative to the bottom of the tank is referenced. This happens twice in each final equation. The offset distance between the bottom of the tank and the outlet orifice is a_1 and a_2 for tank 1 and 2 respectively. This offsetting of dimensions does not apply for the distance from the bottom of the orifice to the datum as this distance is 0.

$$A_1 \cdot \frac{dh_1}{dt} + a_1 = -A_{out1} \cdot C_{d1} \cdot \sqrt{2 \cdot g \cdot (0 - (h_1 + a_1))} \quad (3.3)$$

$$A_2 \cdot \frac{dh_2}{dt} + a_2 = -A_{out2} \cdot C_{d2} \cdot \sqrt{2 \cdot g \cdot (0 - (h_2 + a_2))} \quad (3.4)$$

To solve the equations, integrate both sides.

$$\int_{h_{o1}}^{h_1(t)} \frac{dh_1}{\sqrt{h_1 + a_1}} = \int_0^t \frac{-A_{out1} \cdot C_{d1} \cdot \sqrt{2 \cdot g}}{2 \cdot A_1} dt - a_1 \quad (3.5)$$

$$\int_{h_{o2}}^{h_2(t)} \frac{dh_2}{\sqrt{h_2 + a_2}} = \int_0^t \frac{-2 \cdot C_{d2} \cdot \sqrt{2 \cdot g}}{2 \cdot A_2} dt - a_2 \quad (3.6)$$

The result of this integration is used to compute the Cd values for each tank. The values from these equations are essential to ensuring that the double tank model reflects the gathered data as accurately as possible. h_{o1} and h_{o2} are the heights from the bottom of each respective tank.

$$h_t(t) = (\sqrt{h_{o1} + a} - \frac{C_{d1} \cdot A_{out1} \cdot \sqrt{2 \cdot g}}{2 \cdot A_1} \cdot t)^2 - a_1 \quad (3.7)$$

$$h_t(t) = (\sqrt{h_{o2} + a} - \frac{C_{d2} \cdot A_{out2} \cdot \sqrt{2 \cdot g}}{2 \cdot A_2} \cdot t)^2 - a_2 \quad (3.8)$$

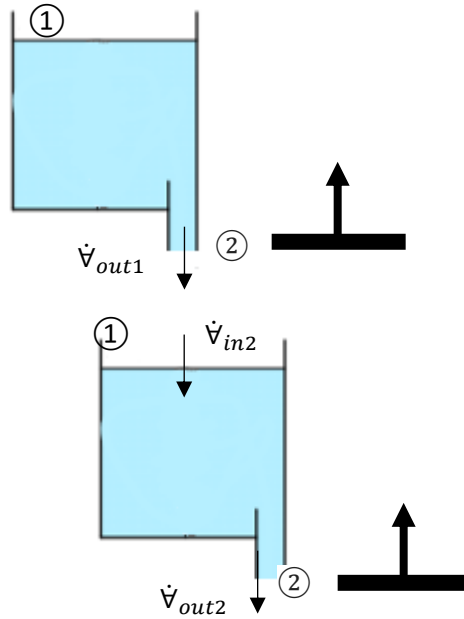


Figure 3: Free body diagram of both tanks.

Once the tanks have been modeled individually the transition to a double tank system is straightforward. Recall equations 1.2 and 2.3 for conservation of mass rate form. Simply equate the new tank 2 inlet volumetric flowrate with the outlet volumetric flowrate from tank 1 and the double tank draining simulation equations have been fully derived. The tank 1 outlet flow rate was derived in equations 3.3 and 3.4.

$$\frac{dh_1}{dt} = - \underbrace{\frac{A_{out1} \cdot C_{d1} \cdot \sqrt{2 \cdot g \cdot (h_1 + a_1)}}{A_1}}_{\dot{V}_{out1}} \quad (4.1)$$

$$\frac{dh_2}{dt} = \dot{V}_{in2} - \underbrace{\frac{A_{out2} \cdot C_{d2} \cdot \sqrt{2 \cdot g \cdot (h_2 + a_2)}}{A_2}}_{\dot{V}_{out2}} \quad (4.2)$$

$$\dot{V}_{in2} = \frac{A_{out1} \cdot C_{d1} \cdot \sqrt{2 \cdot g \cdot (h_1 + a_1)}}{A_1} \quad (4.3)$$

Results and Discussion

The dimensions of the upper and lower tanks and their outlets were provided to us by professors Burchett, Cunningham, and Danesh and recorded in Table 1. The cross-sectional areas of the tanks and corresponding orifices are shown in Table 2.

Table 1. Parameters describing the physical dimensions of each tank

Tank	Tank width (in)	Tank depth (in)	Orifice diameter (in)	Outlet height (in)	Outlet inset (in)
Upper	7.25	3.3125	0.359	3.375	0.5
Lower	7.25	3.3125	0.237	3.375	0.75

Table 2. The cross sectional area of each tank and corresponding orifice in square inches

Tank	Tank Area (in ²)	Orifice Area (in ²)
Upper	24.0156	0.1012
Lower	24.0156	0.0441

As a first step the discharge coefficient values were assumed to be 0.7. Using equations 3.7 and 3.8 and MATLAB's optimization functions the optimum C_d values were calculated. To quantify the difference between the model and the data we used a standard error estimate (SEE)

$$SEE = \sqrt{\frac{\sum_{i=1}^n (x_{model,i} - x_{data,i})^2}{n-2}} \quad (5.1)$$

where n is the number of data points. The optimization was designed to minimize the SEE values. The model was simulated with the original and optimized C_d values and plotted against the data. This is shown in Figures 1 and 2. The discharge coefficients and SEE values for both the initial and optimized model for the upper and lower tank are shown in Table 3.

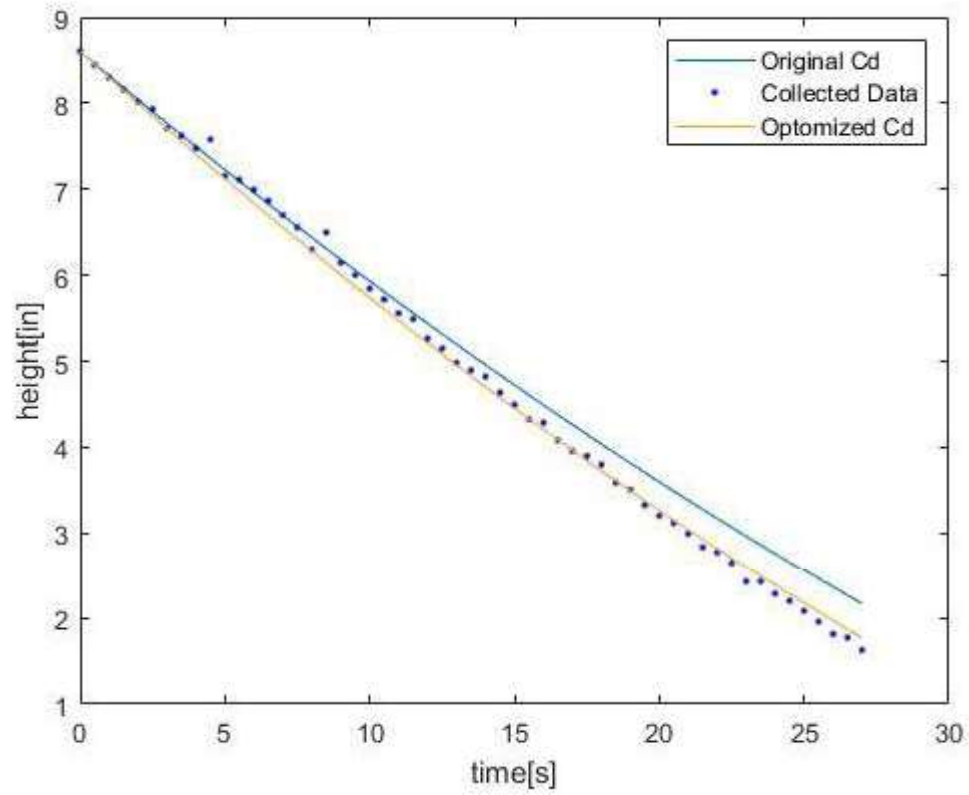


Figure 4. A comparison of the experimental and computed heights measured in inches versus time measured in seconds. Displays the upper tank with C_d set to 0.7 as well as the optimized C_d of 0.75.

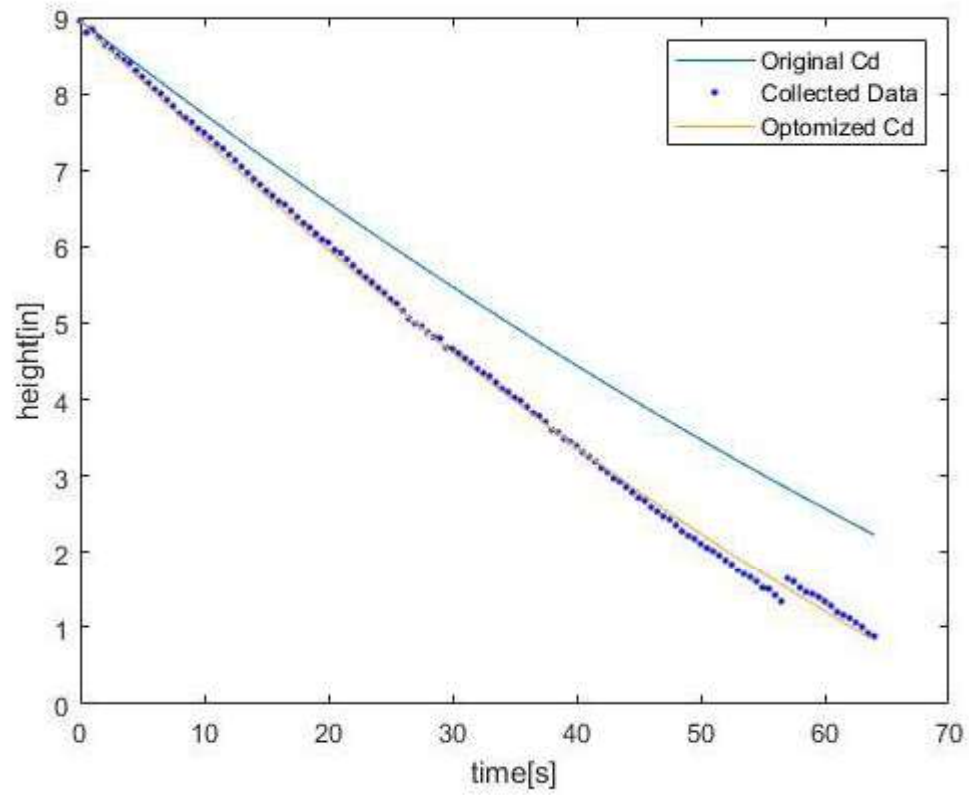


Figure 5. A comparison of the experimental and computed heights measured in inches versus time measured in seconds. Displays the lower tank with C_d set to 0.7 as well as the optimized C_d of 0.89.

Table 3. Initial and optimized discharge coefficients and SEE values for both the upper and lower tank.

	Upper tank		Lower tank	
Case	Discharge coefficient, C_d	SEE (in)	Discharge coefficient, C_d	SEE (in)
Initial model	0.7	0.2878	0.7	0.9436
Optimized model	0.75	0.1100	0.89	0.0863

The experimental data for both tanks was plotted alongside the model data in Figure 3 to visualize how closely our models fit the gathered data. Table 4 also compares the measured and simulated responses for specific times.

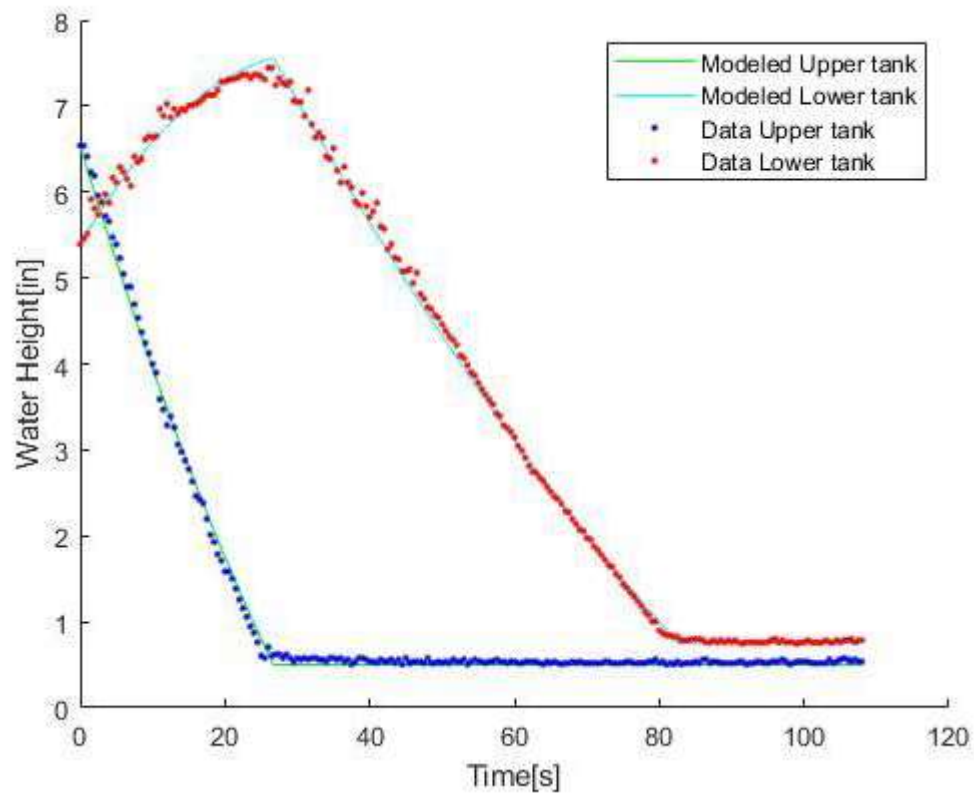


Figure 6. Plot of modeled and gathered data of the two tank systems with optimized C_d 's. Water height is plotted in inches versus Time measured in seconds.

Table 4. Measured and simulated responses for specific times.

Case	Time for upper tank to empty	Time for lower level to drain to a level of 3 inches	Time for the lower tank to drain to a level of 1 inch
Measured	27.0616 s	79.1139 s	60.9824 s
Predicted	26.6809 s	79.9308 s	60.9583 s
Percent Difference	1.417%	1.027%	0.039%

Overall, the model fit the gathered data well. In Table 5, the percent differences between the measured and predicted times were extremely small. The lowest percent difference was between the measured and predicted time for the lower tank to drain to a level of 1 in. at 0.039%. The highest percent difference was between the measured and predicted time for the upper tank to empty at 1.417%. As seen in Figure 3, the model of the upper tank fits closely to the gathered data especially after the water height of the lower tank is constant at about 25 seconds. The model of the upper tank fits less closely than the lower tank especially around the peak of the curve. After about 50 seconds, the model fits the gathered data more closely. The fit between our model and gathered data can be quantitatively described by the calculated SEE values which are 0.1100 and 0.0863 for the upper and lower tank respectively.

The differences between our model and the gathered data can be attributed to several factors related to the physical aspects of the experimental set up as well as the assumptions that we used to derive the equations we used to calculate certain values. With the chosen way that the experiment is set up there are certain losses that occur that affect the model's fit to the data. Loss occurs at the orifice exit due to the fact that the outlets are not perfectly smooth. Since not all the water in the tank will leave the tank due to surface tension at the inner wall height there is also loss there as well. We also assume that the water from the upper tank to the lower tank is instantaneous and without resistance. However, there was a loss of water due to the water spending a certain amount of time in the air when traveling between the two tanks and any splashes that occur when the water contacts the surface of the water in the second tank. This would explain why the discrepancy between the model and the data around the peak of the lower tank is significantly larger than other times on the figure. The loss from the splashing increases as the height of the water increases and gets closer to the actual height of the tank.

There were also several assumptions made to model the two-tank system for calculations that are not entirely accurate. One assumption that we make is that the water used in the experiment is incompressible allowing us to treat density as a constant. Even though clean water is used in each experiment run, water from the Terre Haute area is very heavy which causes dirt and debris to accumulate in the tanks and orifices over time. This means that our assumption that the water is incompressible is not entirely correct as the water is not the "same substance" throughout. This would also contribute to the water losses described earlier. In our model, we assume that the change in height of the water tank level is slow enough that we can also assume that the flow is steady-state. However, if the orifice area becomes large enough or the tank drains fast enough effects from an unsteady flow will start to dominate. This explains why at the times our tank starts to drain faster, at the linear region of the upper tank data and at the peak of the lower tank data, we also see the biggest discrepancy between the model and the data.

These results were used to compare the measured response data to modeled responses to estimate a discharge coefficient for both the upper and lower tank as well as quantitatively and qualitatively see how accurate our estimate was as stated in the lab objective. We were also able

to validate the single-tank model results using results of a two-tank experiment by achieving nearly identical graphical results.

Conclusion and Recommendations

In this lab, the behavior of a two-tank system was effectively modeled through the determination and optimization of drainage coefficients using experimental data provided to the team, and the use of conservation principles and the orifice relationships. Voltage sensor readings were converted to water height data over time.

The standard error estimate was minimized over a centered range of drainage coefficients for each tank using MATLAB, then each coefficient was used to predict the height vs. time relationship for a theoretical system using the same operating parameters as the experiment itself. Through optimization techniques, the group determined a drainage coefficient of 0.75 for the upper tank, and 0.89 for the lower tank which corresponded to standard error values of 0.11 and 0.86 respectively. These coefficients allowed the model fits to very closely predict the experimental data. For the upper tank, the group's drain-time prediction was just 1.4% off of the experimental value, and just 0.039% different from the measured value for the lower tank.

Factors that hindered an error free prediction include non-instantaneous flow between the tanks, splashing of water, imperfections in the water, or losses due to unsteady flow. Some of these factors are absorbed by an adjusted drainage coefficient, but some would lead to inconsistency in the results of the experiment. To isolate our findings from these factors in the lab, the group might have tried to use filtered or distilled water, use new tanks with smoothened outlets, and would have adjusted the position of each tank to minimize splashing.